Spring 2010

1

$P(x) = x^3 - 2x^2 + 3x - 4$

$P(4) = 4^3 - 2(4)^2 + 3(4) - 4 = 64 - 32 + 12 - 4 = 40$

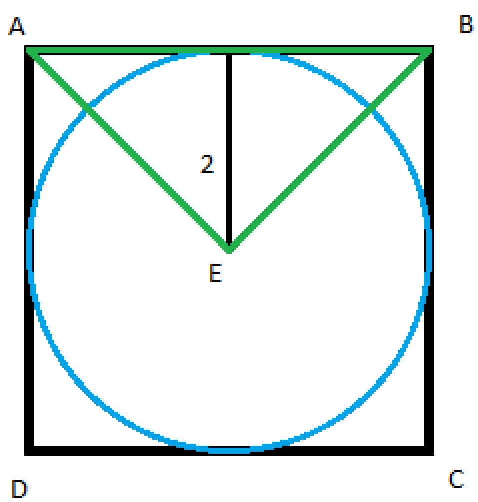
$P(2) = 2^3 - 2(2)^2 + 3(2) - 4 = 8 - 8 + 6 - 4 = 2$

$P(4) - P(2) = 40 - 2 = 38 = 2 * 19$

The largest prime factor of 38 is 19

So the answer is **B. 19**

2



The area of triangle ABE is one fourth the area of the square, and the area of the part of the circle inside triangle ABE is one fourth of the area of the circle. So what we are looking for is $\frac{1}{4}(area\ of\ square - area\ of\ circle)$

The square has sides of length 4, so its area is $4^2 = 16$

The circle has radius of 2, so its area is $\pi 2^2 = 4\pi$

So now we have $\frac{1}{4}(16 - 4\pi) = 4 - \pi$

So the answer is **C.** $\boldsymbol{4 - \pi}$

3

$ax + b = 10, x = 2$

$bx + a = 8, x = 3$

$2a + b = 10$	substitution
$a + 3b = 8$	substitution and reordering
$4a + 2b = 20$	two times the first equation
$5a + 5b = 28$	addition
$a + b = \frac{28}{5}$	divide by 5

So the answer is **B.** $\mathbf{\frac{28}{5}}$

4

$\frac{x+1}{x-3} \geq 2$

Multiplying by x-3 gives two cases

If $x > 3$ (x-3 is positive), $x + 1 \geq 2(x - 3)$

$x + 1 \geq 2x - 6$

$7 \geq x$

$x \leq 7$

In interval form, (3,7]

OR

If $x < 3$ (x-3 is negative), $x + 1 \leq 2(x - 3)$

$x + 1 \leq 2x - 6$

$7 \leq x$

$x \geq 7$

Since x is never less than 3 and greater than or equal to 7, this provides no solution

So the answer is **E. (3,7]**

5 This is an arithmetic sequence, so

$a_0 = 2$

$a_1 = 2 + d$

$a_2 = 2 + 2d$

$a_3 = 2 + 3d = a_1^2 - 8 = (2 + d)^2 - 8 = 4 + 4d + d^2 - 8 = d^2 + 4d - 4$

So $2 + 3d = d^2 + 4d - 4$

$0 = d^2 + d - 6$

$0 = (d - 2)(d + 3)$

$d = 2 \; or - 3$

$a_5 = 2 + 5d > 0$

So d has to be 1 (2+5(2)=12, 2+5(-3)=-13)

$a_3 = 2 + 3(2) = 8$

So the answer is **C. 8**

6

Solving this problem without a calculator is something that I unfortunately do not know how to do. I recommend having a decent graphing calculator for the test (you can borrow a TI 83 Plus from the math lab by leaving your cell phone if you're at Tarrant County College SE, and some are available from the test proctors). For this explanation, and others where I recommend using a calculator, I assume you are using a TI 83 or similar graphing calculator.

We are given $a^3 + b^3 + c^2 = 2010$, and a,b, and c are positive integers. We want this in a $y = f(x)$ form to use in our calculator. I recommend solving for c, because it has the most possible values (we want as little guesswork as possible).]

$a^3 + b^3 + c^2 = 2010$

$c^2 = 2010 - a^3 - b^3$

$c = \sqrt{2010 - a^3 - b^3}$

Unfortunately, our TI 83 doesn't work well with functions of more than one variable (or at all, as well as I can tell), so we have to guess on either a or b. It really doesn't matter which since they do the same thing. We can start our guess at 1, and move up. I take 2 guesses at a time because the TI 83's table shows y1 and y2 on one screen, and scrolling is strange when moving columns.

Y1=√ (2010-1^3-x^3)

Y2=√(2010-2^3-x^3)

Look at the table, and we notice no integer values, so we need to guess again.

Y1=√(2010-3^3-x^3)

Y2=√(2010-4^3-x^3)

Once again, we don't get any integer values…

Y1=√(2010-5^3-x^3)

Y2=√(2010-6^3-x^3)

Looking through the table, we see that at 9, Y1=34. Y1 had us guessing 5, so this means that 5^3+9^3+34^2=2010

We can put it into the calculator to make sure, and it does indeed work, so a=5, b=9, c=34 OR a=9, b=5, c=34

Either way, they ask for a+b which is 5+9 and that is 14

So the answer is **D. 14**

7.

$z = a + bi$

$z^2 = 21 - 20i$

$z^2 = (a + bi)^2 = (a + bi)(a + bi)$	square both sides of first equation
$z^2 = a^2 + abi + abi + b^2i^2 = a^2 + 2abi - b^2$	FOIL and $i^2 = -1$
$21 - 20i = a^2 - b^2 + 2abi$	$z^2 = z^2$
$a^2 - b^2 = 21$	real #'s=real #'s
$2ab = -20$	imaginary #'s =

$ab = -10$	simplify (divide 2)
$a = -\frac{10}{b}$	set up for substitution
$\left(-\frac{10}{b}\right)^2 - b^2 = 21$	substitution.
$\left(-\frac{10}{b}\right)^2 - b^2 - 21 = 0$	set = 0
$\frac{100}{b^2} - b^2 - 21 = 0$	
$100 - b^4 - 21b^2 = 0$	multiply by b^2 (cancel fractions)
$-b^4 - 21b^2 + 100 = 0$	reorder
$b^4 + 21b^2 - 100 = 0$	divide by -1
$(b^2 + 25)(b^2 - 4) = 0$	factor
$b^2 = 4$	$b^2 + 25$ gives imaginary solutions
$b = \pm 2$	
$b = 2$	we only need one solution set
$a(2) = -10$	substitution
$a = -5$	

$|a| + |b| = |-5| + |2| = 5 + 2 = 7$

So the answer is **A. 7**

8

We are given $\frac{AC}{BC} = \frac{BC}{5AB}$ and we need to find $\frac{AC}{BC}$

Since the sizes of AC, AB, and BC don't matter (only their proportions do), we can define them as follows to make the problem easier.

$BC = 1$

This means $AC + 1 = AB$

We can then rewrite the problem as $\frac{AC}{1} = \frac{1}{5(AC+1)}$ and we just need to find AC.

$5AC(AC+1) = 1$	cross-multiply
$5AC^2 + 5AC - 1 = 0$	standard form
$AC = \frac{-5 \pm \sqrt{(-5)^2 - 4(5)(-1)}}{2(5)}$	quadratic formula

$AC = \frac{-5 \pm \sqrt{25+20}}{10}$

$AC = \frac{-5 \pm \sqrt{45}}{10}$

$AC = \frac{-5 \pm 3\sqrt{5}}{10}$

Since AC cannot be it cannot be $\frac{-5-3\sqrt{5}}{10}$

So the answer is **A.** $\boldsymbol{\frac{-5+3\sqrt{5}}{10}}$

9

$\lfloor x \rfloor$ floors x, and we are asked to find $\sum_{n=1}^{2010} \lfloor \log_5 n \rfloor$ (That is, the sum, as n varies from 1 to 2010, of $\lfloor \log_5 n \rfloor$)

If $1 \leq n < 5$, then $0 \leq \log_5 n < 1$, so $\lfloor \log_5 n \rfloor = 0$, This range includes 4 numbers.
Our total in this range $= 4 * 0 = 0$

If $5 \leq n < 25$, then $1 \leq \log_5 n < 2$, so $\lfloor \log_5 n \rfloor = 1$, This range includes 20 numbers.
Our total in this range $= 20 * 1 = 20$

If $25 \leq n < 124$, then $2 \leq \log_5 n < 3$, so$\lfloor \log_5 n \rfloor = 2$. This range includes 100 numbers.
$100 * 2 = 200$

From 125 to 624, $\lfloor \log_5 n \rfloor = 3$. There's 500 in that range.

$500 * 3 = 1500$

And finally, from 625 to 2010, $\lfloor \log_5 n \rfloor = 4$, and there's 1386 of those

$1386 * 4 = 5544$

$5544 + 1500 + 200 + 20 + 0 = 7264$

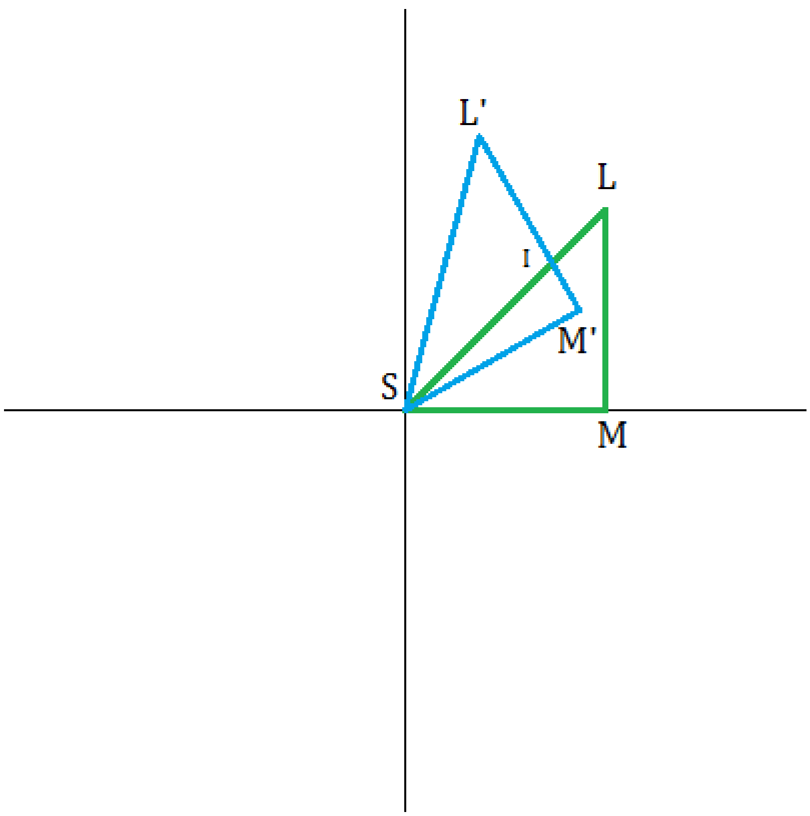
So the answer is **E. 7264**

10

In order for a number to be prime, it must be a product of only itself and one. So only one of the dice can be a number other than one, and it must be prime, so it can be 2, 3, or 5. It can be any of the three dice, so there are 3 combinations * 3 orders = 9 ways to get the desired outcome. There are 6*6*6 = 216 total outcomes, so the probability is $\frac{9}{216} = \frac{1}{24}$

So the answer is **B.** $\mathbf{\frac{1}{24}}$

11



Here is a picture of our triangles. We want the total area of the shape. We can get this by finding the area of ΔSML, doubling it ($\Delta SM'L'$ has the same area), and subtracting the area of $\Delta SM'I$

$\Delta SML = \frac{1}{2}\, 10 * 10 = 50$

$50 * 2 = 100$

$\Delta SM'I$ has a base of 10 units, and an angle of $45° - 30° = 15°$

$\Delta SM'I = \frac{1}{2} * 10 * 10 \sin(15) = 12.94$

$100 - 12.94 = 87.06$

So the answer is **E. 87**

12

Each area is the product of 2 sides. If we call our sides a, b, and c, we have 3 equations

$a * b = 48$

$a * c = 50$

$b * c = 54$

Notice that we don't need to find a, b, or c. All we need is the volume, a*b*c.

$a * b * c = V$

$a^2 * b^2 * c^2 = V^2$

$a * b * a * c * b * c = V^2$

$48 * 50 * 54 = V^2$

$129600 = V^2$

$360 = V$

So the answer is **A. 360**

13

The 16 factors of 2010 don't really matter. They can be split into 8 pairs, each pair multiplying to 2010. Multiplying all of them together gets 2010^8. Let's consider the rows of the magic square (let's call the products of the rows a, b, c, and d). Multiplying the product in each row should get 2010^8 ($a * b * c * d = 2010^8$). Also, the product of each row is the same as the others ($a = b = c = d$). So the product of any row is $2010^2 = 4040100$

$a * a * a * a = 2010^8$

$a^4 = 2010^8$

$a = 2010^2 = 4040100$

Here is one solution of the MMS described. All rows, columns, and diagonals multiply to 4,040,100

30	134	1005	1
201	5	6	670
2	2010	67	15
335	3	10	402

So the answer is **4040100**

14

$$f(x) = \sqrt{\frac{x^2+1}{x^2-1}}$$

$$f\big(f(x)\big) = \sqrt{\frac{\left(\sqrt{\frac{x^2+1}{x^2-1}}\right)^2 +1}{\left(\sqrt{\frac{x^2+1}{x^2-1}}\right)^2 -1}}$$

$$= \sqrt{\frac{\frac{x^2+1}{x^2-1}+1}{\frac{x^2+1}{x^2-1}-1}} = \sqrt{\frac{\frac{x^2+1}{x^2-1}+\frac{x^2-1}{x^2-1}}{\frac{x^2+1}{x^2-1}-\frac{x^2-1}{x^2-1}}} = \sqrt{\frac{\frac{2x^2}{x^2-1}}{\frac{2}{x^2-1}}} = \sqrt{x^2} = |x|$$

Thus, $f^2(x) = |x|$, and $f^3(x) = f(x)$ because all of the x terms in $f(x)$ get squared anyway

Thus $f^{even}(x) = |x|$ and more importantly, $f^{2010}(x) = |x|$

So the answer is **B. |x|**

15

The fastest way I know of to do this problem is just by counting the configurations.

We can have all of the plates be one color, for 3 combinations (red, white, or blue).

```
RRRR
WWWW
BBBB
```

We can have 2 plates each of 2 colors in 2 ways, alternating or grouped. This gives us 6 more combinations.

```
RRWW
RRBB
WWBB
RWRW
RBRB
WBWB
```

We can also have 3 plates of one color and one of another, for another 6 combinations

```
RRRW
RRRB
WWWR
WWWB
BBBR
BBBW
```

There are 3 unique ways to arrange 2 plates of one color, and one of each of the others, with 3 color combinations, for a total of 9 arrangements.

```
RRWB
```

RRBW

RWRB

WWRB

WWBR

WRWB

BBRW

BBRW

BRBW

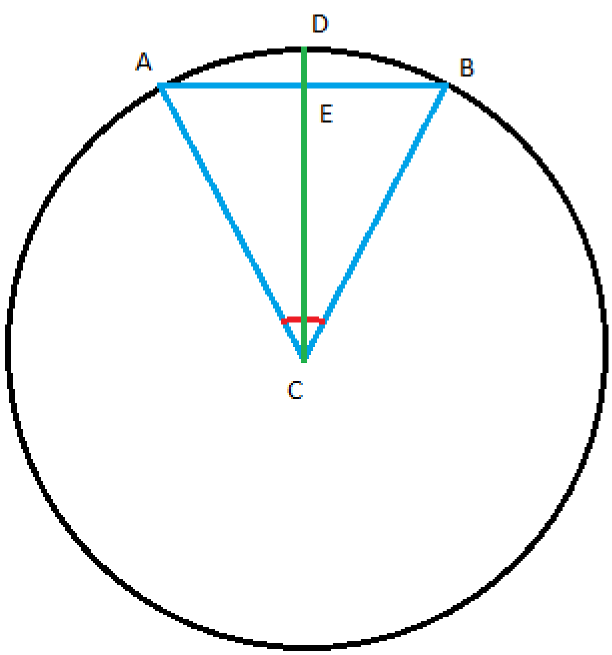
Any other arrangement would be a rotation of one we already have.

So in total we have $3 + 6 + 6 + 9 = 24$ ways of arranging the plates without rotation.

So the answer is **B. 24**

16

We are given a line of length 100, and a circular arc of length 101 connecting it. Let's complete the circle.



$AB = 100, AE = 50, EB = 50, Arc\ AB = 101$, and we need to find DE

We know that $Arc\ AB = BC * \theta$ (BC is the radius)

and $\sin\left(\frac{\theta}{2}\right) = \frac{EB}{BC}$

$101 = BC * \theta$

$BC = \frac{101}{\theta}$

$\sin\left(\frac{\theta}{2}\right) = \frac{50}{\frac{101}{\theta}}$

$\sin\left(\frac{\theta}{2}\right) = \frac{50\theta}{101}$

$\sin\left(\frac{\theta}{2}\right) - \frac{50\theta}{101} = 0$

Now we can use a graphing calculator to get θ. Using numerical methods is beyond the scope of this test.

In a TI-83 (or similar), make sure radians are set, then press y=,
enter sin(x/2)-50x/101
press 2nd, then trace (calc)
select zero.
We want a positive angle, so go slightly right of 0 and press enter for the left bound, go past the sign change and press enter for the right bound, then press enter again for the guess.

This gives us $\theta = .4881933914$

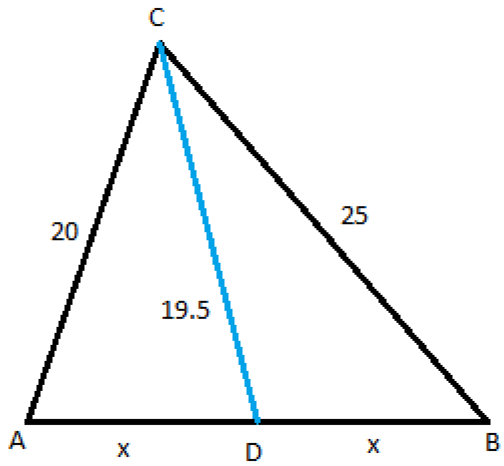
$CD = BC = \frac{101}{\theta} = 206.8852258$

$CE = BC \cos\left(\frac{\theta}{2}\right) = 206.8852258 \cos(.2440966957) = 200.7523266$

The height of the track, $DE = CD - CE = 6.132899$

So the answer is **D. 6**

17



D is the midpoint of AB, so $AD = DB = \frac{1}{2}AB$. Let x represent this length.

Law of cosines: $\cos A = \frac{b^2+c^2-a^2}{2bc}$

For ΔABC, $\cos A = \frac{20^2+(2x)^2-25^2}{2(20)(2x)} = \frac{400-625+4x^2}{80x} = \frac{4x^2-225}{80x}$

For ΔADC, $\cos A = \frac{20^2+x^2-19.5^2}{2(20)x} = \frac{400-380.25+x^2}{40x} = \frac{19.75+x^2}{40x}$

$\frac{4x^2-225}{80x} = \frac{19.75+x^2}{40x}$	cos A = cos A
$40x(4x^2 - 225) = 80x(19.75 + x^2)$	cross multiply
$4x^2 - 225 = 2(19.75 + x^2)$	divide by 40x
$4x^2 - 225 = 39.5 + 2x^2$	distribute
$2x^2 = 264.5$	subtract $2x^2$, add 225.
$x^2 = 132.25$	divide by 2
$x = 11.5$	square root (length is positive)
$AB = 2x = 23$	

So the answer is **B. 23**

18

For this problem it is useful to note that 1001 and 770 are multiples of 7, and 110 is not.

1001 is the first 4-digit palindrome, and is a multiple of 7.

Next are (multiples of 7 in bold)

1001
1111
1221
1331
1441
1551
1661
1771
1881
1991

2/10 are multiples of 7. This is the same for the rest. 2002=1001*2 and is a multiple of 7, add 770 to get 2772, another multiple of 7, the other 8 are not. 3003 and 3773 in the 3000's, 8 not, etc.

$\frac{2}{10} = \frac{1}{5}$

So the answer is **D** $\mathbf{\frac{1}{5}}$

19

$a + b = m$

$a^2 + b^2 = n$

$a^3 + b^3 = m + n$

m and n are positive integers, so let's try to get an equation relating them.

$m^2 = a^2 + 2ab + b^2$

$m^2 - 2ab = a^2 + b^2$

$m^2 - 2ab = n$

$-2ab = n - m^2$

$ab = \frac{m^2-n}{2}$

$(a+b)(a^2 - ab + b^2) = m + n$

$m(n - ab) = m + n$

$n - ab = 1 + \frac{n}{m}$

$-ab = 1 + \frac{n}{m} - n$

$ab = n - \frac{n}{m} - 1$

$\frac{m^2}{2} - \frac{n}{2} = n - \frac{n}{m} - 1$

$m^3 - mn = 2mn - 2n - 2m$

$m^3 + 2m = 3mn - 2n$

$m^3 + 2m = n(3m - 2)$

$n = \frac{m^3+2m}{3m-2}$

Now that we have n as a function of m, we can test values easily (or we can let our TI-83 do the work).

Press Y=, enter (X^3+2X)/(3X-2), then table. This gives us:

x	y	x	y
1	3	9	29.88
2	3	10	36.42857
3	4.714286	11	43.64516
4	7.2	12	51.52941
5	10.38462	13	60.08108
6	14.25	14	69.3
7	18.78947	15	79.18605
8	24	16	89.73913

And we can see that the value of n increases with m, and the highest integer value on the table is 24 (when $m=8$). Since 36 is the highest answer choice, we can be fairly confident that 24 is correct.

(By the way, m is 8 and n is 24 when a and b are $4+2i$ and $4-2i$)

So the answer is **B. 24**

20.

Because you cannot take chips from 2 piles opposite each other, it is impossible to win if the game is symmetrical. Therefore the winning strategy is to make sure the game is symmetrical on your opponents turn, so that they cannot win. We are given

3

4 2

4

So in order to make it symmetrical, only taking from two adjacent piles, we need to take 2 from West and 1 from South, bringing the board to

3

2 2

3

And afterwards, whatever move they make you do the opposite. They will eventually bring half the board to 0, such as

0

X 0

Y

Then you can take the remaining piles and win.

So the answer is **A. 2W 1S**